JOURNAL OF APPROXIMATION THEORY **10**, 227–236 (1974)

Metric Entropy of Some Classes of Sets with Differentiable Boundaries*

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Communicated by G. G. Lorentz

Let $I(k, \alpha, M)$ be the class of all subsets A of R^k whose boundaries are given by functions from the sphere S^{k-1} into R^k with derivatives of order $\leqslant \alpha$, all bounded by M. (The precise definition, for all $\alpha > 0$, involves Hölder conditions.) Let $N_d(\epsilon)$ be the minimum number of sets required to approximate every set in $I(k, \alpha, M)$ within ϵ for the metric d, which is the Hausdorff metric h or the Lebesgue measure of the symmetric difference, d_λ. It is shown that up to factors of lower order of growth, $N_d(\epsilon)$ can be approximated by $\exp(\epsilon^{-r})$ as $\epsilon \downarrow 0$, where $r = (k-1)/\alpha$ if $d = h$ or if $d = d_\lambda$ and $\alpha \geqslant 1$. For $d = d_\lambda$ and $(k-1)/k < \alpha \leqslant 1$, $r \leqslant (k-1)/(k\alpha - k + 1)$. The proof uses results of A. N. Kolmogorov and V. N. Tikhomirov [4].

1. INTRODUCTION

We consider classes of subsets A of R^k whose boundaries ∂A are defined by maps of the sphere S^{k-1} into R^k with bounded derivatives of order $\leqslant\alpha$ for some $\alpha < \infty$. Using Hölder conditions, such classes are defined for all $\alpha > 0$ (not necessarily integral). Given k, α, and a uniform bound M on derivatives of orders $\leqslant\alpha$ (for more detailed definitions see Section 2 below), we have a class $I(k, \alpha, M)$ of subsets of R^k. We ask: given $\epsilon > 0$, how many sets are needed to form an ϵ-dense set in $I(k, \alpha, M)$, i.e., to approximate each set within ϵ, for the Hausdorff metric h or for the metric d_λ which is the Lebesgue measure of the symmetric difference. We find that as $\epsilon \downarrow 0$, the required number $N(\epsilon)$, of sets, is approximated by $\exp(\epsilon^{-r})$ for a suitable exponent r depending on k, α, M and the choice of metric h or d_λ; we write $r = r_h$ or $r = r_\lambda$, respectively. The approximation is proved only in the sense that given $t < r < s$, $\exp(\epsilon^{-t}) < N(\epsilon) < \exp(\epsilon^{-s})$ for ϵ small enough.

* This research was partially supported by National Science Foundation Grant GP-29072.

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Thus $N(\epsilon)$ is asymptotic to $\exp(\epsilon^{-r})$ with suitable factors of lower order of growth inserted, but here we do not find these factors precisely.

Theorem 3.1 below relates r to the given k and α. This result extends previously known theorems on metric entropy of classes of functions (Kolmogorov–Tikhomirov [4, Theorem XV]; Clements [2, Theorem 3]; Lorentz [6, Theorem 10]). The main difficulty in the extension results from the fact that boundaries of sets in $I(k, \alpha, M)$ are not restricted except for differentiability and may intersect themselves in complicated ways. For example, there is a set in $I(2, 17, 1)$ with infinitely many components. We have

$$r_h(I(k, \alpha, M)) = (k-1)/\alpha;$$
$$r_\lambda(I(k, \alpha, M)) = (k-1)/\alpha \qquad \text{if} \quad \alpha \geqslant 1;$$
$$r_\lambda(I(k, \alpha, M)) \leqslant (k-1)/(k\alpha - k + 1) \qquad \text{if} \quad (k-1)/k < \alpha \leqslant 1,$$

where I conjecture that the last inequality for r_λ is also an equality.

The exponent $(k-1)/\alpha$ for classes of functions goes back to Kolmogorov and Tikhomirov [4]. Relations between sets and boundary functions are developed in the preliminary Section 2. The boundary functions on spheres need not be one-to-one.

In Section 4 we consider the class $C(U)$ of all convex closed subsets of any fixed bounded open set $U \subset R^k$. We find

$$r_h(C(U)) = r_\lambda(C(U)) = (1/2)(k-1).$$

Thus convex sets behave like sets with exactly twice differentiable boundaries, as is perhaps not surprising. (On R^1, a convex function f has a second derivative f'' which is a positive Radon measure; even when f'' is a function, it need not satisfy any Hölder condition.) The proof in Section 4, however, uses convexity rather than second derivatives *per se*.

While the results of this paper were found with probabilistic applications in view [3, Theorems 4.2 and 4.3], it seemed appropriate to give them a separate presentation.

2. Preliminaries: Boundaries

Let (S, d) be any metric space. Given $\epsilon > 0$, let $N(S, \epsilon)$ be the smallest number of sets of diameter $\leqslant 2\epsilon$ which cover S. The *exponent of entropy* of S is defined by

$$r(S) = r_d(S) = \limsup_{\epsilon \downarrow 0} [\log \log N(S, \epsilon)]/|\log \epsilon|.$$

(If $r(S) < \infty$, (S, d) must be totally bounded.)

For any two subsets A, B of S, we have the Hausdorff distance $h(A, B)$ defined as follows:

$$h_1(A, B) = \sup_{x \in A} \inf_{y \in B} d(x, y),$$

$$h(A, B) = \max[h_1(A, B), h_1(B, A)].$$

For closed sets, h is a metric. A set $A \subset S$ is called *ϵ-dense* iff $h(A, S) \leqslant \epsilon$.

For measurable subsets of a measure space (S, μ) modulo null sets, there is a metric d_μ defined by $d_\mu(A, B) = \mu(A \triangle B)$, where $A \triangle B$ is the symmetric difference $(A \sim B) \cup (B \sim A)$.

In the following, λ denotes Lebesgue measure. Exponents of entropy for d_λ will be written r_λ.

Now we define spaces of functions on spheres "with bounded derivatives of orders $\leqslant \alpha$" for any $\alpha > 0$. Let β be the greatest integer $< \alpha$ and $\gamma = \alpha - \beta > 0$. For any open set $U \subset R^k$, let $F(U, \alpha)$ be the set of all real functions f on U such that:

(a) the partial derivatives $D^p f = \partial^{|p|} f / \partial x_1^{p_1} \cdots \partial x_k^{p_k}$ exist for $|p| = p_1 + \cdots + p_k \leqslant \beta$;

(b) $\|f\|_\alpha < \infty$ where

$$\|f\|_\alpha = \sup\{| D^p f(x) - D^p f(y)|/| x - y |^\gamma + | D^q f(x)|: \\ | q | \leqslant | p | = \beta, x \neq y \in U, x \in U\}.$$

Let S^{k-1} be the unit sphere in R^k:

$$S^{k-1} = \{x \in R^k: | x |^2 = x_1^2 + \cdots + x_k^2 = 1\}.$$

We can cover S^{k-1} by finitely many coordinate patches V_j so that there are C^∞ isomorphisms $\Phi_j: U \to V_j$ where U is the open ball $\{y: | y | < 1\} \subset R^{k-1}$. We can assume that Φ_j is actually a C^∞ isomorphism from a neighborhood W of the closure of U into S^{k-1}. Then each partial derivative of Φ_j is uniformly bounded and the vectors $\partial \Phi_j / \partial x_i$ for $i = 1,..., k - 1$ are linearly independent on W.

We define $F(V_j, \alpha)$ as the set of all real-valued functions f on V_j such that $f \circ \Phi_j \in F(U, \alpha)$. Let $F(S^{k-1}, \alpha)$ be the set of all real-valued functions f on S^{k-1} such that the restriction of f to V_j is in $F(V_j, \alpha)$ for each j. Then let $\|f\|_\alpha = \sup_j \|f \circ \Phi_j\|_\alpha$. This norm $\| \cdot \|_\alpha$ depends on the choice of V_j and Φ_j but is topologically equivalent to the norms defined by other allowed choices of V_j and Φ_j.

Taking the k-fold Cartesian product of copies of $F(S^{k-1}, \alpha)$, we obtain a Banach space $(F^{(k)}(S^{k-1}, \alpha), \|\cdot\|_\alpha)$ of functions from S^{k-1} into R^k, where $\|(f_1, ..., f_k)\|_\alpha = \max_j \|f_j\|_\alpha$. Now for $M > 0$, let

$$G(k, \alpha, M) = \{f \in F^{(k)}(S^{k-1}, \alpha): \|f\|_\alpha \leqslant M\}.$$

Here we recall a basic definition from algebraic topology. Let f and g be two maps from a topological space S into another space T. Then f and g are called *homotopic* iff there is a continuous F from $[0, 1] \times S$ into T such that $F(0, \cdot) = f$ and $F(1, \cdot) = g$. F is called a *homotopy* of f and g.

Next we shall define an "interior" $I(f)$ for each $f \in G(k, \alpha, M)$, so that, e.g., if f is the identity on S^{k-1}, $I(f)$ is the usual open unit ball. The following definition was kindly suggested to me by J. Munkres.

DEFINITION. For any continuous map f of a topological space S into another space T, let $I(f)$ be the set of all $x \in T \sim \text{range}(f)$ such that in $T \sim \{x\}$, f is not homotopic to any constant map of S into a point $t \in T \sim \{x\}$.

The proof of the following fact was also told me by J. Munkres.

LEMMA 2.1. *Suppose F is a homotopy of f and g. Then* $I(f) \triangle I(g) \subset \text{range } F$.

Proof. Suppose $x \in I(f) \sim I(g)$. If $x \notin \text{range}(F)$, then f and g are homotopic in $T \sim \{x\}$. Clearly homotopy is transitive. Since g is homotopic to a constant map in $T \sim \{x\}$, so is f, a contradiction. The proof is complete.

If f is the identity map of S^{k-1} into R^k, then $I(f)$ is the usual open unit ball by well-known theorems of algebraic topology. Also, if f and g are homotopic in $R^k \sim \{0\}$, then $0 \in I(g)$. Thus the above definition seems broad enough to cover cases of interest.

Let $I(k, \alpha, M) = \{I(f): f \in G(k, \alpha, M)\}$.

3. THE EXPONENTS OF ENTROPY OF $I(k, \alpha, M)$

In the following, I conjecture that equality holds in (3.4). It seems that a proof might require construction of some rather pathological sets.

THEOREM 3.1. *Let* $0 < \alpha < \infty$ *and* $0 < M < \infty$. *Then*

$$r_h(I(k, \alpha, M)) = (k - 1)/\alpha; \tag{3.2}$$

If $\alpha \geqslant 1$, $\quad r_\lambda(I(k, \alpha, M)) = (k - 1)/\alpha;$ (3.3)

If $(k - 1)/k < \alpha \leqslant 1$, $\quad r_\lambda(I(k, \alpha, M)) \leqslant (k - 1)/(k\alpha - k + 1).$ (3.4)

If $0 < \alpha < 1$, $r_\lambda(I(k, \alpha, M)) \geqslant (k - 1)/\alpha$.

Proof. Let $F(U, \alpha, \gamma) = \{f \in F(U, \alpha): \|f\|_\alpha \leqslant \gamma\}$, using the definitions in Section 2. Kolmogorov and Tikhomirov [4, Sect. 5, Theorems XIII–XV] have shown that for any bounded open $U \subset R^k$ and $0 < \zeta < \infty$, $r_s F(U, \alpha, \zeta) = (k - 1)/\alpha$ where s is the supremum metric, $s(f, g) = \sup\{|f(x) - g(x)|\}$. By definition of $G(k, \alpha, M)$ it follows that $r_s G(k, \alpha, M) \leqslant (k - 1)/\alpha$, proving $r_h(I(k, \alpha, M)) \leqslant (k - 1)/\alpha$.

Now suppose $f, g \in G(k, \alpha, M)$ and $s(f, g) \leqslant \epsilon$, where $\epsilon > 0$. Let $F(t, x) \equiv (1 - t)f(x) + tg(x)$ for $0 \leqslant t \leqslant 1$, $x \in S^{k-1}$. By Lemma 2.1, $d_\lambda(I(f), I(g)) \leqslant \lambda(\text{range } F)$.

If $\alpha \geqslant 1$, the maps in $G(k, \alpha, M)$ are uniformly Lipschitzian. Thus $\lambda(\text{range } F) = O(\epsilon)$ as $\epsilon \downarrow 0$, uniformly for $f \in G(k, \alpha, M)$. Hence $r_\lambda(I(k, \alpha, M)) \leqslant (k - 1)/\alpha$.

Next let $(k - 1)/k < \alpha \leqslant 1$. There is a $K < \infty$ such that for $0 < \delta \leqslant 1$, there is a set $E_\delta \subset S^{k-1}$ such that for all $x \in S^{k-1}$, $|x - y| \leqslant \delta^{1/\alpha}$ for some $x \in E_\delta$, where E_δ has at most $K\delta^{(1-k)/\alpha}$ elements. Then for any $f \in G(k, \alpha, M)$ and $z \in \text{range} f$ there is an $x \in E_\delta$ with $|f(x) - z| \leqslant N\delta$ for some $N > M$.

Let c_k be the volume of the unit ball in R^k. Given $\epsilon > 0$ let

$$\delta = [\epsilon / K c_k 4^k N^k]^{\alpha/(k\alpha - k + 1)}.$$

Then $\lambda\{x : \exists y : |f(y) - x| < 3N\delta\} \leqslant 4^k N^k K c_k \delta^{(k\alpha - k + 1)/\alpha} = \epsilon$ if $\delta \leqslant 1$, as is true for ϵ small enough. To obtain a $3N\delta$-dense set in $G(k, \alpha, M)$ it suffices to approximate functions within $N\delta$ at each point of E_δ. Hence for ϵ small,

$$\begin{aligned} N(I(k, \alpha, M), \epsilon, d_\lambda) &\leqslant \exp\{K\delta^{(1-k)/\alpha} \log[(2k + 1)^k / \delta^k]\} \\ &\leqslant \exp\{C_k \epsilon^{(1-k)/(k\alpha - k + 1)} \,|\log \epsilon|\} \end{aligned}$$

for some constant C_k, so (3.4) follows.

To prove $\geqslant$ and hence equality in (3.2) and (3.3) we use the following fact, due to G. F. Clements [2, Theorem 3]. The proof here is different and seems simpler.

LEMMA 3.5 (Clements). *Let V be a bounded open set in R^{k-1}, $k \geqslant 2$, $\alpha > 0$, and $0 < \gamma < \infty$. Then $r_1(F(V, \alpha, \gamma)) \geqslant (k - 1)/\alpha$ where r_1 is the exponent of entropy for the L^1 metric $d_1(f, g) = \int_V |f - g| \, d\lambda$.*

Proof. We can assume V is the open cube $\{x : 0 < |x_j| < 1, j = 1,..., k - 1\}$. Let f be a positive C^∞ function with support in V. Let $\|f\|_\alpha = N < \infty$. For $Q \geqslant 1$ and $t \in R^{k-1}$ let $g(x) = f(Qx + t)$. Then for some $Z < \infty$, $\|g\|_\alpha \leqslant ZQ^\alpha$ for all $Q \geqslant 1$.

For each positive integer Q there exist Q^{k-1} such functions g_j with disjoint support, $j = 1,..., Q^{k-1}$. For each set $A \subset \{1,..., Q^{k-1}\}$, let $g_A = \sum_{j \in A} g_j$. We shall show that there are many such sets A, different in many places. This type of result seems to be known, but the following proof seems short enough to include, and I know no explicit references for the result.

LEMMA 3.6. *For any positive integer n and any set B with n elements, there is a collection of sets $E_i = E(i) \subset B$, $i = 1,\ldots, m$, such that $m \geq e^{n/8}$ and such that for $i \neq j$, $E_i \triangle E_j$ has at least $n/5$ elements.*

Proof. Given any set $E \subset B$, the number of sets $F \subset A$ such that $E \triangle F$ has at most $n/5$ elements is $2^n B(n/5, n, 1/2)$ where $B(r, n, p)$ is the probability of at most r successes in n independent trials with probability p of success in each trial. According to Kolmogorov's exponential bound [5, p. 254],

$$B(n/5, n, 1/2) \leqslant \exp(-.126n) < \exp(-n/8).$$

Thus we can inductively choose the sets E_i with $m \geqslant e^{n/8}$, proving Lemma 3.6.

Now the functions $h_A \equiv \gamma g_A/Q^\alpha Z$ all belong to $F(V, \alpha, \gamma)$. Let $\kappa = \int |f| \, d\lambda > 0$. Then for $i \neq j$,

$$\int | h_{E(i)} - h_{E(j)} | \, d\lambda \geqslant Q^{k-1}\gamma\kappa/5ZQ^{k-1+\alpha} = \gamma\kappa/5ZQ^\alpha.$$

Let $\epsilon = \gamma\kappa/5ZQ^\alpha$. Then Q is proportional to $\epsilon^{-1/\alpha}$. Letting $Q \to \infty$ and applying Lemma 3.6 yields, for some constant $\beta > 0$,

$$N(F(V, \alpha, \gamma), \epsilon) \geqslant \exp\{\beta\epsilon^{(1-k)/\alpha}\}.$$

Thus Lemma 3.5 is proved.

There is a one-to-one C^∞ map $G = (G_1, \ldots, G_k)$ of S^{k-1} into R^k with a flat face. Here "flat face" means there is an open set $U \subset S^{k-1}$ such that $G_1(U) = \{0\}$, and for some $\delta > 0$ and all t such that $|t| < \delta$ and $x \in U$, $G(x) + (t, 0,\ldots, 0) \in I(G)$ iff $t > 0$. Let $H = (G_2, \ldots, G_k)$. Then $H(U)$ is an open set $V \subset R^{k-1}$. For some $M_0 < \infty$, $G \in G(k, \alpha, M_0)$. Given any $M > 0$, we can replace G by a small multiple of itself and assume $M_0 < M/2$. We can also assume $V = \kappa C$ where $\kappa > 0$ and C is the open unit cube in R^{k-1}. Then for some small enough $\zeta > 0$, with $\zeta < \delta$, all the following functions $\varphi_A \in G(k, \alpha, M)$:

$$\begin{aligned} \varphi_A(x) &= G(x) \qquad \text{for} \quad x \notin U \\ &= G(x) + (\zeta h_A(H(x)/\kappa), 0,\ldots, 0) \qquad \text{for} \quad x \in U, \end{aligned}$$

where h_A is as in the proof of Lemma 3.5, with $\gamma \leqslant \min(1, M_0)$.

For any sets A and $B \subset \{1,\ldots, Q^{k-1}\}$,

$$d_\lambda(I(\varphi_A), I(\varphi_B)) = \zeta \int_V | h_A - h_B | \, d\lambda,$$

for Q large enough. Thus by Lemma 3.5 and its proof, we have equality in (3.2) and (3.3) for all $M > 0$ and Theorem 3.1 is proved.

4. Convex Sets

Let $C(U)$ denote the class of all convex closed subsets of U. It turns out that the exponent of entropy of $C(U)$, for U bounded, is $(1/2)(k-1)$ although second derivatives of boundaries of polyhedra in $C(U)$ are only measures, not functions.

Theorem 4.1. *Let U be a bounded open set in R^k. Then $r_\lambda(C(U)) = r_h(C(U)) = (1/2)(k-1)$.*

Proof. We choose a fixed point $\zeta \in U$. Let $s = h(U, \{\zeta\})$. We have for any $C, D \in C(U)$ by [1, p. 41, 5]:

$$d_\lambda(C, D) \leqslant 2c_k[-s^k + (s + h(C, D))^k] \leqslant Nh(C, D) \tag{4.2}$$

where N depends on k and s but not on C, D. Thus to prove $r(C(U)) \leqslant (1/2)(k-1)$ we need only consider the Hausdorff metric.

Lemma 4.3. *Suppose given vectors x, y, u, v in R^k such that $(x-y, u) \geqslant 0$ and $(x-y, v) \leqslant 0$. Then*

$$|x + u - y - v| \geqslant \max(|x-y|, |u-v|).$$

Proof.

$$\begin{aligned}|x + u - y - v|^2 &= |x-y|^2 + |u-v|^2 + 2(x-y, u-v)\\ &\geqslant |x-y|^2 + |u-v|^2.\end{aligned}$$

Q.E.D.

A convex set C will be called *analytic* iff there is an entire analytic function f such that $C = \{x \in R^k : f(x) \leqslant 1\}$, and the gradient of f is nonzero on the boundary ∂C. It is known that analytic convex sets are h-dense in the class of all bounded convex sets [1, pp. 36–37]. If C is analytic and $p \in \partial C$, let $\varphi(p) = \operatorname{grad} f(p)/|\operatorname{grad} f(p)|$. Then φ is a continuous 1–1 map of ∂C onto S^{k-1}. Let $e(p, q)$ be the (smallest nonnegative) angle between $\varphi(p)$ and $\varphi(q)$. Then $0 \leqslant e(p, q) \leqslant \pi$. Let $d(p, q) = |p - q|$.

Lemma 4.4. *Given a bounded open $U \subset R^k$, there is an $M < \infty$ such that whenever $0 < \delta < 1$, and C is any analytic convex subset of U, there is a set $A \subset \partial C$ with* $\operatorname{card}(A) \leqslant M\delta^{1-k}$ *such that A is δ-dense in ∂C for $d + e$.*

Proof. Let B be a fixed ball such that $x + y \in B$ whenever $x \in U$ and $|y| \leqslant 1$. Then there is a constant $S < \infty$ such that whenever $0 < \epsilon < 1$ there is an ϵ-dense set $B_\epsilon \subset \partial B$ with $\operatorname{card}(B_\epsilon) \leqslant S\epsilon^{1-k}$.

Let C be convex and analytic, $C \subset U$. Then for every $p \in \partial B$, there is a unique nearest point $n(p) \in \partial C$, with $|p - n(p)| \geq 1$. The function $n(\cdot)$ maps ∂B 1–1 onto ∂C. Suppose $q \in \partial B$ and $|p - q| < \epsilon$. Let $u = p - n(p)$, $v = q - n(q)$. Then we can apply Lemma 4.3 with $x = n(p)$ and $y = n(q)$ to conclude $|n(p) - n(q)| < \epsilon$ and $|u - v| < \epsilon$. Let θ be the angle between u and v, so that $e(n(p), n(q)) = \theta$. Let $u_1 = u/|u|$, $v_1 = v/|v|$. Since $|u| \geq 1$ and $|v| \geq 1$, we have $|u_1 - v_1| < \epsilon$. Also $|u_1 - v_1| = 2\sin(\theta/2)$. We know $\theta \leq \pi \sin(\theta/2)$ for $0 \leq \theta \leq \pi$ by concavity. Thus $e(n(p), n(q)) \leq \pi\epsilon/2 < 2\epsilon$. Hence we can let $M = 2^k S$, $A = \{n(p): p \in B_\epsilon\}$, proving Lemma 4.4.

LEMMA 4.5. *Let C be an analytic convex set and $0 < \delta \leq \pi/4$. Let A be a δ-dense set in ∂C for $d + e$. Let C_A be the intersection of all half-spaces which include C and are bounded by hyperplanes supporting C (tangent to ∂C) at points of A. Then $h(C, C_A) \leq 2\delta^2$.*

Proof. Clearly $C_A \supset C$. Conversely let $x \in \partial C$ and choose $y \in A$ with $(d + e)(x, y) \leq \delta$. Let T_x be the tangent hyperplane to ∂C at x. Let u be the unit outward normal vector to ∂C and T_x at x. Then $x + \gamma u \in T_y$ for some $\gamma > 0$. To maximize γ, we may assume $y \in T_x$ (this particular argument does not use analyticity). Now $\gamma \leq \delta \tan \delta \leq 2\delta^2$ since $\tan \theta \leq 2\theta$ for $0 \leq \theta \leq \pi/4$. For every $z \in C_A$ there is a nearest point $x \in C$, and $|z - x| \leq 2\delta^2$. Q.E.D.

Proof of Theorem 4.1. First we prove $r_h(C(U)) \leq (1/2)(k - 1)$. We can assume U is a cube. Let t be the diameter of U. We may assume $t \geq 2$. There is an $N < \infty$ such that $N \geq 1$ and whenever $0 < \epsilon \leq \pi/4$ there is an $\epsilon/2$-dense set $U_\epsilon \subset B$ with $\text{card}(U_\epsilon) \leq N\epsilon^{-k}$ (where B is a fixed large ball $\supset U$ as in Lemma 4.4), and such that there is a $\tan^{-1}(\epsilon/3t)$-dense set $V_\epsilon \subset S^{k-1}$ for the angular metric e with $\text{card}(V_\epsilon) \leq N\epsilon^{1-k}$.

Let W_ϵ be the set of all convex polyhedra $P \subset U$ formed by intersections of at most $M\epsilon^{(1-k)/2}$ half-spaces H_j (here M is as in Lemma 4.4) such that each hyperplane ∂H_j contains a point of U_ϵ and is orthogonal to a vector v in V_ϵ, and v is directed outward from H_j. Then

$$\text{card}(W_\epsilon) \leq \exp\{[M\epsilon^{(1-k)/2}] \log[N^2\epsilon^{1-2k}]\}.$$

Hence

$$\limsup_{\epsilon \downarrow 0}(\log \log \text{card } W_\epsilon)/|\log \epsilon| \leq (1/2)(k - 1).$$

Now we show that W_ϵ is 12ϵ-dense in $C(U)$ for h. To approximate a set $C \in C(U)$, we may assume C is analytic. We take the set $A \subset \partial C$ provided by Lemma 4.4 for $\delta = \epsilon^{1/2}$. At each $x \in A$ let T_x be the tangent hyperplane to ∂C. Let v_x be the unit outward normal vector at x. Choose $p_x \in U_\epsilon$ with

$|p_x - x - \epsilon v_x| \leqslant \epsilon/2$. Let J_x be a hyperplane passing through p_x, orthogonal to a vector in V_ϵ, and forming an angle with T_x less than $\tan^{-1}(\epsilon/3t)$. Let H_x be the half-space on the side of T_x containing x. Then $H_x \supset C$ since $h(\{p_x\}, C) \geqslant \epsilon/2$ and $(t + \epsilon/2)(\epsilon/3t) \leqslant \epsilon/2$. Let $C_\epsilon = \bigcap_{x \in A} H_x \supset C$.

Now take any $y \in \partial C$ and v_y as above. Take $x \in A$ such that $(d + e)(x, y) < \epsilon^{1/2}$. Then $|y - p_x| < 3\epsilon^{1/2}$ while T_y and T_x form an angle less than $2\epsilon^{1/2}$. We have $x \in C$ and $y \in H_x$. As in the proof of Lemma 4.5, it follows that $y + \gamma v_y \notin C_\epsilon$ for $\gamma \geqslant 12\epsilon$, so that $h(C, C_\epsilon) \leqslant 12\epsilon$. Since $C_\epsilon \in W_\epsilon$, we have proved $r(C(U)) \leqslant (1/2)(k - 1)$.

For the converse inequality, by (4.2) it suffices to consider the metric d_λ.

There is a $c > 0$ such that whenever $0 < \epsilon < 1$, there is a set $A_\epsilon \subset S^{k-1}$ with $\text{card}(A_\epsilon) \geqslant c\epsilon^{1-k}$ such that $|x - y| \geqslant 4\epsilon$ for any distinct x and y in A_ϵ. For each $x \in A_\epsilon$, let C_x be the solid spherical cap cut from the unit ball $B_1 = \{y : |y| \leqslant 1\}$ by the hyperplane orthogonal to x and passing through $(1 - \epsilon^2/2)x$. For some constant $\alpha_k > 0$, $\lambda(C_x) \geqslant \alpha_k \epsilon^{k+1}$.

The caps C_x are disjoint. For an arbitrary set $E \subset A_\epsilon$, let

$$D_E = B_1 \sim \bigcup_{x \in E} C_x .$$

Each D_E is convex. We have $h(D_E, D_F) = \epsilon^2/2$ for $E \neq F$ so the proof is easily completed for h. For d_λ we apply Lemma 3.6; taking the sets $E_i = E(i)$ for A_ϵ, we have

$$\lambda(D_{E(i)} \triangle D_{E(j)}) \geqslant \alpha_k \epsilon^{k+1} c \epsilon^{1-k}/5 = \beta_k \epsilon^2$$

for some constant $\beta_k > 0$. Letting $\delta = \beta_k \epsilon^2/3$ we have

$$N(C(U), \delta) \geqslant \exp\{-\gamma_k \delta^{(1-k)/2}\}$$

for some constant $\gamma_k > 0$. Letting $\delta \downarrow 0$, Theorem 4.1 is proved.

Acknowledgment

I thank J. Munkres for showing me the relation between sets and boundaries developed in §2.

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